

NEW VERSIONS OF SCHUR-WEYL DUALITY

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ABSTRACT. After reviewing classical Schur-Weyl duality, we present some other contexts which enjoy similar features, relating to Brauer algebras and classical groups.

1. CLASSICAL SCHUR-WEYL DUALITY

1.1. Schur's double-centralizer result. Consider the vector space $V = \mathbb{C}^n$. The symmetric group \mathfrak{S}_r acts naturally on its r -fold tensor power $V^{\otimes r}$, by permuting the tensor positions. This action obviously commutes with the natural action of $\mathbf{GL}_n = \mathbf{GL}_n(\mathbb{C})$, acting by matrix multiplication in each tensor position. So we have a $\mathbb{C}\mathbf{GL}_n$ - $\mathbb{C}\mathfrak{S}_r$ bimodule structure on $V^{\otimes r}$. (Here $\mathbb{C}G$ denotes the group algebra of a group G .) In 1927, Schur [Sc] proved that the image of each group algebra under its representation equals the full centralizer algebra for the other action. More precisely, if we name the representations as follows

$$(1) \quad \mathbb{C}\mathbf{GL}_n \xrightarrow{\rho} \text{End}(V^{\otimes r}) \xleftarrow{\sigma} \mathbb{C}\mathfrak{S}_r$$

then we have equalities

$$(2) \quad \rho(\mathbb{C}\mathbf{GL}_n) = \text{End}_{\mathfrak{S}_r}(V^{\otimes r})$$

$$(3) \quad \sigma(\mathbb{C}\mathfrak{S}_r) = \text{End}_{\mathbf{GL}_n}(V^{\otimes r}).$$

(Here, for a given set S operating on a vector space T through linear endomorphisms, $\text{End}_S(T)$ denotes the set of linear endomorphisms of T commuting with each endomorphism coming from S .)

Results of Carter-Lusztig [CL] and J.A. Green [G] (and others) show that all the above statements remain true if one replaces \mathbb{C} by an arbitrary infinite field K .

These notes are based on a lecture, various versions of which I have given in the past year, in a number of locations, including Stuttgart, Birmingham, Queen Mary (London), Lancaster, Manchester, Oxford, and Cambridge. I'm grateful to the organizers of those events for the opportunity to present these ideas.

1.2. Schur algebras. The finite-dimensional algebra in (2) above, for any K , is known as the *Schur algebra*, and often denoted by $S_K(n, r)$ or simply $S(n, r)$. The Schur algebra “sees” the part of the rational representation theory of the algebraic group $\mathrm{GL}_n(K)$ occurring (in some appropriate sense) in $V^{\otimes r}$. More precisely, there is an equivalence between r -homogeneous polynomial representations of $\mathrm{GL}_n(K)$ and $S_K(n, r)$ -modules. In characteristic 0, those representations (as r varies) determine all finite-dimensional rational representations, while in positive characteristic they still provide a tremendous amount of information.

The representation σ in (1) is faithful if $n \geq r$, so σ induces an isomorphism

$$(4) \quad K\mathfrak{S}_r \simeq \mathrm{End}_{\mathrm{GL}_n}(V^{\otimes r}) = \mathrm{End}_{S_K(n, r)}(V^{\otimes r}) \quad (n \geq r).$$

This leads to intimate connections between polynomial representations of $\mathrm{GL}_n(K)$ and representations of $K\mathfrak{S}_r$, a theme that has been exploited by many authors in recent years. Perhaps the most dramatic example of this is the result of Erdmann [E] (building on previous work of Donkin [Do3] and Ringel [R]) which shows that knowing decomposition numbers for all symmetric groups in positive characteristic will determine the decomposition numbers for general linear groups in the same characteristic. Conversely, James [Ja] had already shown that the decomposition matrix for a symmetric group is a submatrix of the decomposition matrix for an appropriate Schur algebra. Thus the (still open) general problem of determining the modular characters of symmetric groups is equivalent to the similar problem for general linear groups (over infinite fields).

1.3. The enveloping algebra approach. Return to the basic setup, over \mathbb{C} . One may differentiate the action of the Lie group $\mathrm{GL}_n(\mathbb{C})$ to obtain an action of its Lie algebra \mathfrak{gl}_n . Replacing the representation ρ in (1) by its derivative representation $d\rho : \mathfrak{U}(\mathfrak{gl}_n) \rightarrow \mathrm{End}(V^{\otimes r})$ leads to the following alternative statement of Schur’s result:

$$(5) \quad d\rho(\mathfrak{U}(\mathfrak{gl}_n)) = \mathrm{End}_{\mathfrak{S}_r}(V^{\otimes r})$$

$$(6) \quad \sigma(\mathbb{C}\mathfrak{S}_r) = \mathrm{End}_{\mathfrak{gl}_n}(V^{\otimes r}).$$

In particular, the Schur algebra (over \mathbb{C}) is a homomorphic image of $\mathfrak{U}(\mathfrak{gl}_n)$. All of this works over an arbitrary integral domain K if we replace $\mathfrak{U}(\mathfrak{gl}_n)$ by its “hyperalgebra” $\mathfrak{U}_K := K \otimes_{\mathbb{Z}} \mathfrak{U}_{\mathbb{Z}}$ obtained by change of ring from a suitable \mathbb{Z} -form of $\mathfrak{U}(\mathfrak{gl}_n)$; see [Do1]. (One can adapt the Kostant \mathbb{Z} -form, originally defined for the enveloping algebra of a *semisimple* Lie algebra, to the *reductive* \mathfrak{gl}_n .)

1.4. The quantum case. Jimbo [Ji] extended the results of 1.3 to the quantum case (where the quantum parameter is not a root of unity). One needs to replace \mathfrak{S}_r by the Iwahori-Hecke algebra $\mathbf{H}(\mathfrak{S}_r)$ and replace $\mathfrak{U}(\mathfrak{gl}_n)$ by the quantized enveloping algebra $\mathbf{U}(\mathfrak{gl}_n)$. The analogue of the Schur algebra in this context is known as the q -Schur algebra, often denoted by $\mathbf{S}(n, r)$ or $\mathbf{S}_q(n, r)$. Dipper and James [DJ] have shown that q -Schur algebras are fundamental for the modular representation theory of *finite* general linear groups.

As many authors have observed, the picture in 1.1 can also be quantized. For that one needs a suitable quantization of the coordinate algebra of the algebraic group \mathbf{GL}_n .

There is a completely different (geometric) construction of q -Schur algebras given in [BLM].

1.5. Integral forms. The Schur algebras $S_{\mathbb{C}}(n, r)$ admit an integral form $S_{\mathbb{Z}}(n, r)$ such that $S_K(n, r) \simeq K \otimes_{\mathbb{Z}} S_{\mathbb{Z}}(n, r)$ for any field K . In fact $S_{\mathbb{Z}}(n, r)$ is simply the image of $\mathfrak{U}_{\mathbb{Z}}$ (see 1.3) under the surjective homomorphism $\mathfrak{U}(\mathfrak{gl}_n) \rightarrow S_{\mathbb{C}}(n, r)$. Similarly, the quantum Schur algebra $\mathbf{S}_{\mathbb{Q}(v)}(n, r)$ admits an integral form defining all specializations via base change. One needs to replace \mathbb{Z} by $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$; then the integral form $\mathbf{S}_{\mathcal{A}}(n, r)$ is the image of the Lusztig \mathcal{A} -form $\mathbf{U}_{\mathcal{A}}$ under the surjection $\mathbf{U}(\mathfrak{gl}_n) \rightarrow \mathbf{S}_{\mathbb{Q}(v)}(n, r)$. (To match this up with various specializations in the literature, one often has to take $q = v^2$.)

1.6. Generators and relations. Recently, in joint work with Giacinto (see [DG]), a very simple set of elements generating the kernel of the surjection $\mathfrak{U}(\mathfrak{gl}_n) \rightarrow S_{\mathbb{C}}(n, r)$ was found. A very similar set of elements generates the kernel of the surjection $\mathbf{U}(\mathfrak{gl}_n) \rightarrow \mathbf{S}_{\mathbb{Q}(v)}(n, r)$. These elements are expressible entirely in terms of the Chevalley generators for the zero part of $\mathfrak{U}(\mathfrak{gl}_n)$ or $\mathbf{U}(\mathfrak{gl}_n)$. Thus we obtain a presentation of $S_{\mathbb{C}}(n, r)$ and $\mathbf{S}_{\mathbb{Q}(v)}(n, r)$ by generators and relations, compatible with the usual Serre (Drinfeld-Jimbo) presentation of $\mathfrak{U}(\mathfrak{gl}_n)$ (resp., $\mathbf{U}(\mathfrak{gl}_n)$). As a result, we find a certain subset of the integral PBW-basis for $\mathfrak{U}(\mathfrak{gl}_n)$ or $\mathbf{U}(\mathfrak{gl}_n)$ the image of which gives an integral basis for $S_{\mathbb{Z}}(n, r)$ or $\mathbf{S}_{\mathcal{A}}(n, r)$. This basis yields a similar basis in any specialization. Moreover, a subset of it provides a new integral basis of $\mathbf{H}(\mathfrak{S}_n)$.

2. THE BRAUER ALGEBRA

From now on I will assume, unless stated otherwise, that the underlying field is \mathbb{C} (it could just as well be any field of characteristic zero).

One expects that many statements will be valid over an arbitrary infinite field, via some appropriate integral form, similar to what happens in type A .

2.1. The algebra $\mathcal{B}_r^{(x)}$. Let R be a commutative ring, and consider the free $R[x]$ -module $\mathcal{B}_r^{(x)}$ with basis consisting of all r -diagrams. An r -diagram is an (undirected) graph on $2r$ vertices and r edges such that each vertex is incident to precisely one edge. One usually thinks of the vertices as arranged in two rows of r each, the *top* and *bottom* rows. (See Figure 1.) Edges connecting two vertices in the same row (different rows) are called *horizontal* (resp., *vertical*). We can compose two such diagrams D_1, D_2 by identifying the bottom row of vertices in the first diagram with the top row of vertices in the second diagram. The result is a graph with a certain number, $\delta(D_1, D_2)$, of interior loops. After removing the interior loops and the identified vertices, retaining the edges and remaining vertices, we obtain a new r -diagram $D_1 \circ D_2$, the composite diagram.

Multiplication of r -diagrams is defined by the rule

$$D_1 \cdot D_2 = x^{\delta(D_1, D_2)}(D_1 \circ D_2).$$

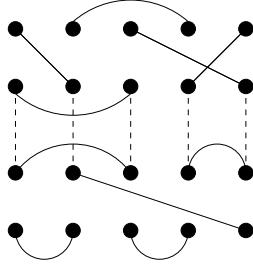
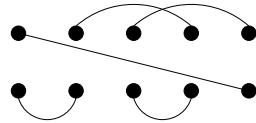
One can check that this multiplication makes $\mathcal{B}_r^{(x)}$ into an associative algebra; this is the Brauer algebra. (See Figures 1–3 for an illustration of the multiplication in the Brauer algebra.)

Note that if we take $x = 1$ then the set of r -diagrams is a monoid under diagram composition, and $\mathcal{B}_r^{(1)}$ is simply the semigroup algebra of that monoid.



FIGURE 1. Two Brauer diagrams D_1, D_2 for $r = 5$.

For any x the group algebra $R[x]\mathfrak{S}_r$ may be identified with the subalgebra of $\mathcal{B}_r^{(x)}$ spanned by the diagrams containing only vertical edges. Such Brauer diagrams provide a graphical depiction of permutations. The group algebra $R[x]\mathfrak{S}_r$ of \mathfrak{S}_r also appears as a quotient of $\mathcal{B}_r^{(x)}$, the quotient by the two-sided ideal spanned by all diagrams containing at least one horizontal edge.

FIGURE 2. Computing the composite of D_1 and D_2 .FIGURE 3. The composite diagram $D_1 \circ D_2$.

Label the vertices in each row of an r -diagram by the indices $1, \dots, r$. For any $1 \leq i \neq j \leq r$ let $c_{i,j}$ be the r -diagram with horizontal edges connecting vertices i, j on the top and bottom rows. All other vertices in the diagram $c_{i,j}$ are vertical, connecting vertex k on the top and bottom rows, for all $k \neq i, j$. Brauer observed that $\mathcal{B}_r^{(x)}$ is generated by the permutation diagrams together with just one of the $c_{i,j}$.

2.2. Schur-Weyl duality. Brauer [Br] introduced the algebra $\mathcal{B}_r^{(x)}$ in 1936 to describe the invariants of symplectic and orthogonal groups acting on $V^{\otimes r}$. (Brauer's conventions were slightly different; we are here following the approach of Hanlon and Wales [HW], who pointed out that $\mathcal{B}_r^{(-n)}$ is isomorphic with the algebra defined by Brauer to deal with the symplectic case.) Let G be Sp_n or O_n , where n is even in the first instance. By restricting the action ρ considered in 1.1 we have an action of G on $V^{\otimes r}$. One can extend the action of \mathfrak{S}_r to an action of $\mathcal{B}_r^{(\epsilon n)}$ (over \mathbb{C}) on $V^{\otimes r}$, where $\epsilon = -1$ if $G = \mathsf{Sp}_n$ and $\epsilon = 1$ if $G = \mathsf{O}_n$. To do this, it is enough to specify the action of the diagram $c_{i,j}$. This acts on $V^{\otimes r}$ as one of Weyl's *contraction* maps contracting in tensor positions i and j . So we have (commuting) representations

$$(7) \quad \mathbb{C}G \xrightarrow{\rho} \text{End}(V^{\otimes r}) \xleftarrow{\sigma} \mathcal{B}_r^{(\epsilon n)}$$

which satisfy Schur-Weyl duality; i.e., the image of each representation equals the full centralizer algebra of the other action:

$$(8) \quad \rho(\mathbb{C}G) = \text{End}_{\mathcal{B}_r^{(\epsilon n)}}(V^{\otimes r})$$

$$(9) \quad \sigma(\mathcal{B}_r^{(\epsilon n)}) = \text{End}_G(V^{\otimes r}).$$

The algebras in equality (8) are the symplectic and orthogonal Schur algebras (see [Do2], [D1], [D2]).

If $n \geq r - 1$ the representation σ in (7) is faithful [Bro]; thus it induces an isomorphism $\mathcal{B}_r^{(\epsilon n)} \simeq \text{End}_G(V^{\otimes r})$.

2.3. Schur-Weyl duality in type D . In type $D_{n/2}$ (n even) the orthogonal group O_n is not connected, and contains the connected semisimple group SO_n (special orthogonal group) as subgroup of index 2. In order to handle this situation, Brauer (see also [Gr]) defined a larger algebra $\mathcal{D}_r^{(n)}$, spanned by the usual r -diagrams previously defined, together with certain *partial* r -diagrams on $2r$ vertices and $r - n$ edges, in which n vertices in each of the top and bottom rows are not incident to any edge, and showed that the action of $\mathcal{B}_r^{(n)}$ can be extended to an action of this larger algebra $\mathcal{D}_r^{(n)}$ on $V^{\otimes r}$. Thus we have representations

$$(10) \quad \mathbb{C}\text{SO}_n \xrightarrow{\rho} \text{End}(V^{\otimes r}) \xleftarrow{\sigma} \mathcal{D}_r^{(\epsilon n)}.$$

Brauer showed that the actions of SO_n and $\mathcal{D}_r^{(n)}$ on $V^{\otimes r}$ satisfy Schur-Weyl duality:

$$(11) \quad \rho(\mathbb{C}\text{SO}_n) = \text{End}_{\mathcal{D}_r^{(\epsilon n)}}(V^{\otimes r})$$

$$(12) \quad \sigma(\mathcal{D}_r^{(\epsilon n)}) = \text{End}_{\text{SO}_n}(V^{\otimes r}).$$

The algebra in (11) is a second Schur algebra in type D , a proper subalgebra of the algebra $\text{End}_{\mathcal{B}_r^{(n)}}(V^{\otimes r})$ appearing in (8) above.

2.4. Generators and relations. One can formulate the above statements of Schur-Weyl duality using enveloping algebras, analogous to 1.3. This leads to a presentation (see [DGS]) of the symplectic and orthogonal Schur algebras which is compatible with (a slight modification of) the usual Serre presentation of the enveloping algebra $\mathfrak{U}(\mathfrak{g})$, where $\mathfrak{g} = \mathfrak{sp}_n$ (n even) or \mathfrak{so}_n .

2.5. The quantum case. There is a q -version of the Schur-Weyl duality considered in this section, although not as developed as in type A . One needs to replace the Brauer algebra by its q -analogue, the Birman-Murakami-Wenzl (BMW) algebra (see [BW], [Mu]), and replace the

enveloping algebra by a suitable quantized enveloping algebra. One can think of the BMW algebra in terms of Kauffman's tangle monoid; see [Ka], [HR], [MW]. (Roughly speaking, tangles are replacements for Brauer diagrams, in which one keeps track of under and over crossings, subject to certain natural relations.) There are applications of the BMW algebra to knot theory, as one might imagine.

This leads to a q -analogue of the symplectic Schur algebras, in particular, which have been studied by Oehms [Oe].

To the best of my knowledge, a q -analogue of the larger algebra $\mathcal{D}_r^{(n)}$ (n even) considered in 2.3 remains to be formulated.

3. THE WALLED BRAUER ALGEBRA

3.1. The algebra $\mathcal{B}_{r,s}^{(x)}$. This algebra was defined in 1994 in [BCHLLS]. It is the subalgebra of $\mathcal{B}_{r+s}^{(x)}$ spanned by the set of (r, s) -diagrams. By definition, an (r, s) -diagram is an $(r + s)$ -diagram in which we imagine a wall separating the first r from the last s columns of vertices, such that:

- (a) all horizontal edges cross the wall;
- (b) no vertical edges cross the wall.

An edge crosses the wall if its two vertices lie on opposite sides of the wall. The multiplication in $\mathcal{B}_{r,s}^{(x)}$ is that of $\mathcal{B}_{r+s}^{(x)}$.

Label the vertices on the top and bottom rows of an (r, s) -diagram by the numbers $1, \dots, r$ to the left of the wall and $-1, \dots, -s$ to the right of the wall. Let $c_{i,-j}$ ($1 \leq i \leq r$; $1 \leq j \leq s$) be the diagram with a horizontal edge connecting vertices i and $-j$ on the top row and the same on the bottom row, and with all other edges connecting vertex k ($k \neq i, -j$) in the top and bottom rows. It is easy to see that the walled Brauer algebra is generated by the permutations it contains along with just one of the $c_{i,-j}$. (Note that $c_{i,-j}$ is the $(r + s)$ -diagram denoted by $c_{i,r+j}$ in 2.1.)

3.2. Dimension. What is the dimension of $\mathcal{B}_{r,s}^{(x)}$? One way to answer that question is to consider the map, *flip*, from $(r + s)$ -diagrams to $(r + s)$ -diagrams, defined by interchanging the top and bottom vertices to the right of the imaginary wall. For example, Figure 4 shows a $(4, 2)$ -diagram (to the left) and its corresponding 6-diagram, obtained from the left diagram by applying *flip*. Note that *flip* is involuntary: applying it twice gives the original diagram back again.

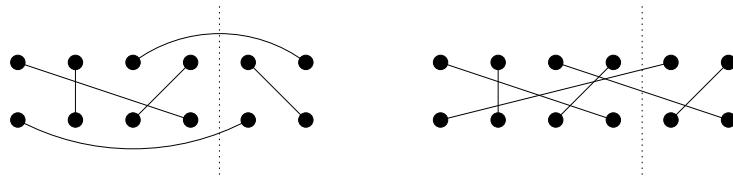


FIGURE 4. A $(4, 2)$ -diagram and its corresponding permutation, after applying *flip*.

One easily checks that the map *flip* carries (r, s) -diagrams bijectively onto the set of $(r + s)$ -diagrams with all edges vertical. Such diagrams correspond with permutations of $r + s$ objects, so the dimension of $\mathcal{B}_{r,s}^{(x)}$ is $(r + s)!$.

3.3. Another view of $\mathcal{B}_{r,s}^{(x)}$. The above correspondence between (r, s) -diagrams and permutations gives another way to think of the multiplication in $\mathcal{B}_{r,s}^{(x)}$. Given two (r, s) -diagrams D_1, D_2 let D'_1, D'_2 be their corresponding permutations obtained by applying *flip*. Define a new (rather bizarre) composition on permutations as follows. Given any two permutation diagrams D'_1, D'_2 (with $r + s$ columns of vertices) identify the first r vertices of the bottom row of D'_1 with the first r vertices of the top row of D'_2 , and identify the last s vertices of the top row of D'_1 with the last s vertices of the bottom row of D'_2 . After removing loops and identified vertices this gives a new permutation diagram D'_3 in which the vertices in the top (resp., bottom) row are the remaining top (bottom) row vertices from the original diagrams.

Let $\delta(D'_1, D'_2)$ be the number of loops removed in computing the composite permutation diagram D'_3 . Define multiplication of permutation diagrams by the rule

$$D'_1 \cdot D'_2 = x^{\delta(D'_1, D'_2)} D'_3$$

In other words, we are multiplying permutations by composing maps “on the right” on one side of the wall, and “on the left” on the other side (roughly speaking). For example, Figure 5 below shows the computation of the composite diagram in the walled Brauer algebra (left column) and the computation in terms of the corresponding permutations (right column). Figure 6 shows the resulting diagrams after the single loop and identified vertices have been removed.

One can check that D'_3 coincides with $(D_1 \circ D_2)'$ and $\delta(D_1, D_2) = \delta(D'_1, D'_2)$. In other words, *flip* defines an algebra isomorphism between

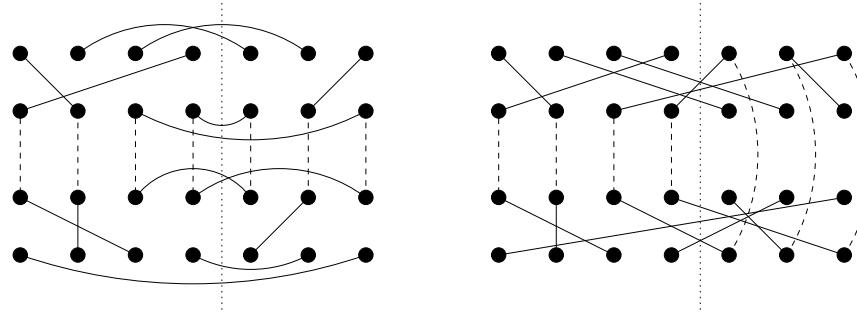


FIGURE 5. Composition of diagrams and permutations. The diagrams on the left correspond under *flip* with the permutations on the right.

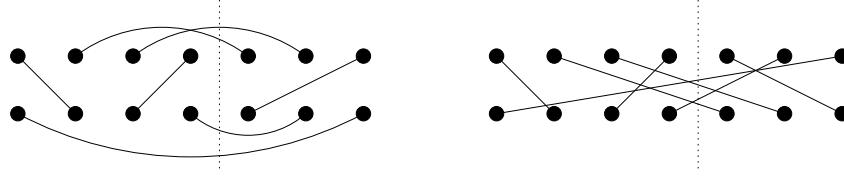


FIGURE 6. The corresponding diagrams resulting from Figure 5. The two diagrams correspond under *flip*.

the algebra $\mathcal{B}_{r,s}^{(x)}$ and the algebra $\tilde{\mathcal{B}}_{r,s}^{(x)}$ spanned by permutation diagrams with the multiplication defined above. Note that in particular $\tilde{\mathcal{B}}_{r,0}^{(x)} \simeq R[x]\mathfrak{S}_r$ and $\tilde{\mathcal{B}}_{0,s}^{(x)} \simeq (R[x]\mathfrak{S}_s)^{\text{opp}}$.

3.4. Schur-Weyl duality. Consider mixed tensor space $V^{r,s} := V^{\otimes r} \otimes V^{*\otimes s}$, where V^* is the usual linear dual space of V . Mixed tensor space is naturally a module for GL_n , and one obtains an action of $\mathcal{B}_{r,s}^{(n)}$ on $V^{r,s}$ simply by restricting the action of $\mathcal{B}_{r+s}^{(n)}$, which acts the same on $V^{r,s}$ as it does on $V^{\otimes(r+s)}$, since on restriction to O_n we have $V \simeq V^*$. Thus we have the following commutative diagram

$$\begin{array}{ccccc}
 \mathbb{C}\text{GL}_n & \xrightarrow{\rho} & \text{End}(V^{r,s}) & \xleftarrow{\sigma} & \mathcal{B}_{r,s}^{(n)} \\
 (13) \quad \downarrow \iota & & & & \downarrow \iota' \\
 \mathbb{C}\text{O}_n & \xrightarrow{\rho} & \text{End}(V^{\otimes(r+s)}) & \xleftarrow{\sigma} & \mathcal{B}_{r+s}^{(n)}
 \end{array}$$

in which the vertical maps ι, ι' are inclusion. By [BCHLLS], the actions of GL_n and $\mathcal{B}_{r,s}^{(n)}$ on $V^{r,s}$ in the first row of the diagram satisfy Schur-Weyl duality:

$$(14) \quad \rho(\mathbb{C}\mathsf{GL}_n) = \text{End}_{\mathcal{B}_{r,s}^{(n)}}(V^{r,s})$$

$$(15) \quad \sigma(\mathcal{B}_{r,s}^{(n)}) = \text{End}_{\mathsf{GL}_n}(V^{r,s}).$$

The algebra in (14) is another Schur algebra $S(n; r, s)$ in type A , studied in [DD]. These Schur algebras provide us with a new family of quasihereditary algebras, generalizing the classical Schur algebras, since $S(n; r, 0) \simeq S(n, r)$. In fact, the $S(n; r, s)$ provide a new class of generalized Schur algebras in the sense of Donkin [Do1]. For fixed n , the family of $S(n; r, s)$ -modules as r, s vary constitutes the family of all rational representations of GL_n . Whence the name *rational* Schur algebras for the $S(n; r, s)$.

When $n \geq r + s$, the representation σ in the top row of (13) above is faithful, so induces an isomorphism $\mathcal{B}_{r,s}^{(n)} \simeq \text{End}_{\mathsf{GL}_n}(V^{r,s})$.

3.5. The quantum case. Quantizations of the walled Brauer algebra have been defined and studied in work of Halverson [Ha], Leduc [Le], Kosuda-Murakami [KM], and Kosuda [K].

4. THE DERANGED ALGEBRA

4.1. The problem. One might wonder if there are versions of Schur-Weyl duality in which the natural module V is replaced by some other representation. Perhaps the first choice would be to replace V with the adjoint module, *i.e.*, an algebraic group acting on its Lie algebra via the adjoint representation. The simplest instance of this would be type A , where we consider the module $\mathfrak{sl}_n^{\otimes r}$ as an SL_n -module, and ask for its centralizer algebra $\text{End}_{\mathsf{SL}_n}(\mathfrak{sl}_n^{\otimes r}) = \text{End}_{\mathsf{GL}_n}(\mathfrak{sl}_n^{\otimes r})$. (It makes no difference whether we regard \mathfrak{sl}_n as module for SL_n or for GL_n . We look at $\mathfrak{sl}_n^{\otimes r}$ rather than $\mathfrak{gl}_n^{\otimes r}$ since \mathfrak{gl}_n is not simple as a GL_n -module or SL_n -module.)

4.2. Relation with Brauer algebras. Even though this is a question about type A , its solution is intimately connected with the walled Brauer algebra. Here is a brief outline of the solution to this problem, recently obtained in [BD]. The main idea is to utilize the decomposition $\mathfrak{gl}_n = \mathfrak{sl}_n \oplus \mathbb{C}$ (as SL_n or GL_n module) to write

$$(16) \quad \mathfrak{gl}_n^{\otimes r} = \mathfrak{sl}_n^{\otimes r} \oplus \bigoplus_{0 \leq t < r} \mathfrak{sl}_n^{\otimes t}$$

where the left-hand-side identifies with $V^{r,r} = V^{\otimes r} \otimes V^{*\otimes r}$ via the natural isomorphism $V \otimes V^* \simeq \text{End}(V) \simeq \mathfrak{gl}_n$. Thus it follows that our desired tensor space $\mathfrak{sl}_n^{\otimes r}$ is isomorphic with a direct summand (as a GL_n or SL_n -module) of the mixed tensor space $V^{r,r}$, and for $n \geq 2r$ the centralizing algebra $\mathcal{C} = \text{End}_{\text{GL}_n}(\mathfrak{sl}_n^{\otimes r})$ is obtainable as a certain subalgebra $e \text{End}_{\text{GL}_n}(V^{r,r})e$ where e is the idempotent corresponding to the projection onto the summand $\mathfrak{sl}_n^{\otimes r}$. Thus

$$(17) \quad \mathcal{C} \simeq e\mathcal{B}_{r,r}^{(n)}e \quad (n \geq 2r)$$

where $e = \prod_i (1 - n^{-1}c_{i,-i})$ (notation of 3.1). The algebra in (17) has a basis consisting of all elements of the form eDe as D ranges over the set of (r,r) -diagrams with no horizontal edges connecting i to $-i$. Such diagrams correspond under *flip*, after inverting the signs labeling the bottom row, with *derangements* of $2r$ objects. (A derangement is a permutation having no fixed points.) Let $N(k) =$ the number of derangements of k objects, then we have by the Inclusion-Exclusion Principle (see [St, 2.2.1 Example])

$$(18) \quad N(k) = \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j!$$

which is the nearest integer to $k!/\text{e}$ ($\text{e} = 2.7182818\dots$). Thus the dimension of the centralizer algebra $e\mathcal{B}_{r,r}^{(n)}e$ is given by $N(2r)$. Because of this connection with derangements, the algebra $e\mathcal{B}_{r,r}^{(n)}e$ is known as the *deranged algebra*.

In [BD] explicit formulas are obtained for the number of times a given simple module L appears as a summand of $\mathfrak{sl}_n^{\otimes r}$. In particular, it is shown that

$$N(r) = \text{the multiplicity of the trivial module in } \mathfrak{sl}_n^{\otimes r}$$

and

$$N(r-1) = \text{the multiplicity of } \mathfrak{sl}_n \text{ in } \mathfrak{sl}_n^{\otimes r}$$

demonstrating that the combinatorics of derangement numbers is inherent in this theory.

4.3. Schur-Weyl duality. At least when $n \geq 2r$, the actions of GL_n and \mathcal{C} on $\mathfrak{sl}_n^{\otimes r}$ satisfy Schur-Weyl duality:

$$(19) \quad \rho(\mathbb{C}\text{GL}_n) = \text{End}_{e\mathcal{B}_{r,r}^{(n)}e}(\mathfrak{sl}_n^{\otimes r})$$

$$(20) \quad \sigma(e\mathcal{B}_{r,r}^{(n)}e) = \text{End}_{\text{GL}_n}(\mathfrak{sl}_n^{\otimes r}).$$

This will almost certainly hold for all n, r .

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